Assignment 8

Hand in no 5, 6, and 7 in November 14.

1. Consider the problem of minimizing $f(x, y, z) = (x + 1)^2 + y^2 + z^2$ subjecting to the constraint $g(x, y, z) = z^2 - x^2 - y^2 - 1,\ z > 0$. First solve it by eliminating z and then by Lagrange multipliers.

2. Let $f, g_1, \cdots, g_m$ be C^1-functions defined in some open U in $\mathbb{R}^{n+m}$. Suppose (x_0, y_0) is a local extremum of f in $\{(x, y) \in U :\ g_1(x, y) = \cdots = g_m(x, y) = 0\ \}$. Assuming that $D_y G(x_0, y_0)$ is invertible where $G = (g_1, \cdots, g_m)$, show that there are $\lambda_1, \cdots, \lambda_m$ such that
$$\nabla f + \lambda_1 \nabla g + \cdots + \lambda_m \nabla g_m = 0\ ,$$
at (x_0, y_0).

3. Let $f \in C(R)$ where R is a closed rectangle. Suppose x solves $x' = f(t, x)$ for $t \in (a, b)$ with $(t, x(t)) \in R$. Show that x can be extended to be a solution in $[a, b]$.

4. Let $f \in C(R)$ where R is a closed rectangle satisfy a Lipschitz condition in R. Suppose that x solves $x' = f(t, x)$ for $t \in [a, b]$ where $(b, x(b))$ lies in the interior of R. Show that there is some $\delta > 0$ such that x can be extended as a solution in $[a, b + \delta]$.

5. Let $D = (a, b) \times \mathbb{R}$ and $f \in C(\overline{D})$ satisfy a Lipschitz condition. Let x be a maximal solution to the (IVP) $x' = f(t, x),\ x(t_0) = x_0,\ t_0 \in (a, b)$ over the maximal interval (α, β). Show that if $\beta < b$, $x(t) \to \infty$ or $x(t) \to -\infty$ as $t \uparrow \beta$.

6. Let f and g be two continuous functions in $\overline{D}$ both satisfying a Lipschitz condition and $f < g$ everywhere. Let x and y be the respective solutions to the (IVP) of f and g satisfying $x(t_0) < y(t_0)$. Show that $x(t) < y(t), t \geq t_0$, as long as they exist.

7. Let $D = \mathbb{R}^2$ and $f \in C(\mathbb{R}^2)$ satisfy a Lipschitz condition. Suppose that $|f(t, x)| \leq C(1+|x|)$ everywhere. Show that all maximal solutions exists on $(-\infty, \infty)$. Hint: Use the previous two questions.

8. Provide a proof to Theorem 3.15 (Picard-Lindelof theorem for systems).